THE SPECTRUM OF PERTURBATIONS AND CONVECTIVE INSTABILITY OF A PLANE, HORIZONTAL FLUID LAYER WITH PERMEABLE BOUNDARIES

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D.D. SHVARTSBLAT
(Perm')

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Convective stability of a fluid heated from below is usually studied under the assumption that the walls of the cavity are impermeable to the fluid and, that there is no flow of fluid across the boundaries. Meanwhile, injection and removal by suction of fluid through the permeable boundaries may exert a decisive influence on the conditions governing the onset of convection and serve as one of the means of controlling the convective instability. The problem, therefore, represents some interest.

Below we consider a plane, horizontal, infinite layer of viscous fluid bounded by two permeable planes kept at different temperatures. Between the planes, a stationary, transverse fluid motion with a uniform vertical velocity, takes place. This represents a generalization of a well known Rayleigh problem on the stability of a plane horizontal fluid layer heated from below to the case, when a transverse motion takes place between the permeable boundaries of the layer.

We obtain the decremental spectra of small, normal, velocity and temperature perturbations and study the convective stability of the fluid. From these decremental spectra we obtain the critical Rayleigh numbers, which depend on the Peclet number characterizing the rate of injection of the fluid. In particular, we find that the transverse flow in the layer leads to increase in the values of the critical Rayleigh number, i.e. to an increase in the convective stability of the fluid. The Bubnov-Galerkin method is used in computing and a large number of the basis functions was utilized.

1. Let us consider a plane, horizontal, infinite layer of a viscous incompressible fluid, bounded by two planes $z = \pm h$ and heated to different temperatures $\mp\theta$ (z-axis is directed vertically upwards, x- and y-axes are horizontal and the coordinate origin is situated in the middle of the layer). We assume that a homogeneous inward flow of fluid takes place with the velocity v_0 through the surface $z = -h$, and the same type of flow in the outward direction, across the surface $z = h$. In this case we have, within the fluid layer, a steady transverse flow of fluid with the following uniform velocity

$$v_x = 0, \quad v_y = 0, \quad v_z = v_0 \tag{1.1}$$

To find the corresponding steady temperature distribution $T_0 = T_0(z)$ we shall write the equation of heat conductivity in its dimensionless form, taking the semi-width h of the layer as the unit distance and θ as the unit temperature

$$aT_0' = T_0'', \qquad a = \frac{v_0 h}{\chi} \tag{1.2}$$

Here a is the Peclet number defined by the semi-width of the layer and the velocity of the steady flow, and χ is the coefficient of thermal diffusivity.

Taking the boundary conditions for the temperature

$$T_0(-1) = 1, \qquad T_0(1) = -1 \tag{1.3}$$

into account we find, from (1.2), the temperature distribution in the layer in the presence of a steady transverse flow of fluid

$$T_0 = \frac{1}{\operatorname{sh} a}(\operatorname{ch} a - e^{az}) \tag{1.4}$$

In the absence of a transverse flow ($a = 0$), (1.4) yields $T_0 = -z$, i.e. a linear temperature distribution in the vertical direction, corresponding to the fluid layer at rest. When the velocity of the transverse flow increases, i.e. when Peclet number a increases, the temperature distribution pattern "displaces" towards the upper ($a > 0$) or the lower ($a < 0$) boundary. At large values of a, a temperature boundary layer of the thickness $1/a$, appears near the boundary.

2. We shall begin our investigation of the convective stability of a fluid layer by writing down the equations governing small perturbations of the steady velocity and temperature distribution. Eliminating from the convection equations the pressure and the x- and y- velocity components in the usual manner, we obtain the following equations for the vertical perturbation velocity component $v_z(x, y, z, t)$ and for the temperature perturbations $T(x, y, z, t)$

$$\begin{aligned} &\frac{\partial}{\partial t}\Delta v_z + \frac{a}{P}\frac{\partial}{\partial z}\Delta v_z = \Delta^2 v_z + \Delta_1 T \qquad \left(P = \frac{\nu}{\chi}\right) \\ &P\frac{\partial T}{\partial t} + a\frac{\partial T}{\partial z} + RT_0' v_z = \Delta T \qquad \left(R = \frac{g\beta\theta h^3}{\nu\chi}\right) \end{aligned} \tag{2.1}$$

Here Δ and Δ_1 denote the three- and two-dimensional Laplacian respectively; β, ν and χ are the respective coefficients of thermal expansion, kinematic viscosity and thermal diffusivity and g is the acceleration due to gravity. Eqs. (2.1) are written in the dimensionless form and the units of time, velocity and temperature and denoted by h^2/ν, $g\beta\theta h^2/\nu$, and θ respectively (the unit of distance was given previously). Three dimensionless parameters appearing in (2.1) are: the Rayleigh (R), Prandtl (P) and Peclet (a) numbers.

Let us consider a normal perturbation of the form

$$v_z = v(z)\exp[-\lambda t + i(k_1 x + k_2 y)] \qquad T = \theta(z)\exp[-\lambda t + i(k_1 x + k_2 y)] \tag{2.2}$$

Here $\lambda = \lambda_r + i\lambda_i$ is the complex perturbation decrement, while k_1 and k_2 are real wave numbers. Inserting (2.2) into (2.1) we obtain the following Eqs. for the perturbation amplitudes $v(z)$ and $\theta(z)$

$$\begin{aligned} -\lambda(v'' - k^2 v) + \frac{a}{P}(v''' - k^2 v') &= (v^{\text{IV}} - 2k^2 v'' + k^4 v) - k^2\theta \\ -\lambda P\theta + a\theta' + RT_0' v &= \theta'' - k^2\theta \end{aligned} \tag{2.3}$$

The following conditions should hold at the boundaries of the layer

$$v = v' = \theta = 0 \quad \text{when } z = \pm 1 \tag{2.4}$$

The complex decrement λ of the normal perturbations is obtained as an eigenvalue of the boundary value problem (2.3) and (2.4), while the perturbation amplitudes $v(z)$ and $\theta(z)$ are its eigenfunctions.

3. We shall solve the boundary value problem (2.3) and (2.4) using the approximate Galerkin method. For this, we shall approximate the solution as follows

$$v = \sum_{n=0}^{N-1} \alpha_n v_n, \qquad \theta = \sum_{m=0}^{M-1} \beta_m \theta_m \tag{3.1}$$

The amplitudes of the normal velocity and temperature perturbations in the fluid at rest shall be used as the basis functions v_n and θ_m. These basis functions will be solutions of the following boundary value problems

$$\Delta^2 v_n + \mu_n \Delta v_n = 0, \qquad v_n(\pm 1) = v_n'(\pm 1) = 0 \tag{3.2}$$

$$\frac{1}{P}\Delta\theta_m + \nu_m\theta_m = 0, \qquad \theta_m(\pm 1) = 0 \tag{3.3}$$

Functions v_n and θ_m are given in the explicit form in e.g. [1]. Normalizing integrals K_l and J_i are

$$K_l = \int_{-1}^{+1} \theta_l^2\, dz = 1 \tag{3.4}$$

$$J_i = \int_{-1}^{+1} v_i \Delta v_i\, dz = \begin{cases} \mu_i\,[(\mu_i - k^2)^{-1/2} k \,\mathrm{th}\, k\,(1 - k\,\mathrm{th}\, k) - 1], & i = 2S \\ \mu_i\,[(\mu_i - k^2)^{-1/2} k \,\mathrm{cth}\, k\,(1 - k\,\mathrm{cth}\, k) - 1], & i = 2S+1 \end{cases}$$

Inserting v and θ from (3.1) into (2.3), multiplying both Eqs. of (2.3) by v_i and θ_l respectively and integrating in z from -1 to $+1$, we obtain a system of linear homogeneous algebraic equations for the coefficients of the expansions (3.1)

$$\sum_{n=0}^{N-1} \alpha_n \left[(\mu_n - \lambda)\,\delta_{in} + \frac{a}{P} B_{in}\right] + k^2 \sum_{m=0}^{M-1} \beta_m D_{im} = 0 \quad (i = 0, 1, 2, \ldots, N-1)$$

$$R \sum_{n=0}^{N-1} \alpha_n C_{ln} + \sum_{m=0}^{M-1} \beta_m\,[(\nu_m - P\lambda)\,\delta_{lm} + aE_{lm}] = 0 \quad (l = 0, 1, 2, \ldots, M-1) \tag{3.5}$$

Matrix elements B_{in} and E_{lm} are given by

$$B_{in} = \frac{1}{J_i}\int_{-1}^{+1} v_i v_n'\, dz, \qquad E_{lm} = \frac{1}{K_l}\int_{-1}^{+1} \theta_l\theta_m'\, dz \tag{3.6}$$

When both indices have the same sign, $B_{in} = E_{lm} = 0$. Otherwise we have

$$B_{in} = \frac{1}{J_i}\frac{2\mu_i\mu_n}{\mu_n - \mu_i}, \qquad E_{lm} = (-1)^{\frac{l+m+1}{2}} \frac{2(l+1)(m+1)}{(l-m)(l+m+2)} \tag{3.7}$$

Matrix elements D_{im} and C_{ln} are given by

$$D_{lm} = \frac{1}{J_i}\int_{-1}^{+1} v_i\theta_m\, dz, \quad C_{ln} = \frac{1}{K_l}\int_{-1}^{+1} T_0'\theta_l v_n\, dz \tag{3.8}$$

$D_{im} = 0$ when i and m differ in sign. If both indices are even, or both odd, we have

$$D_{im} = (-1)^{\frac{m}{2}} \frac{1}{J_i}\frac{2\rho_m\mu_i}{P\nu_m(\mu_i - P\nu_m)}, \quad D_{im} = (-1)^{\frac{m-1}{2}} \frac{1}{J_i}\frac{2\rho_m\mu_i}{P\nu_m(\mu_i - P\nu_m)} \tag{3.9}$$

respectively. When both indices are even, we have

$$C_{ln} = q\,[r^{-1}(f - b) + t^{-1}(b - d)]\, 2\rho_l\,(-1)^{1/2 l} \tag{3.10}$$

while when both are odd, we have

$$C_{ln} = q\,[r^{-1}(c - f) + t^{-1}(d - c)]\, 2\rho_l\,(-1)^{1/2(l+1)} \tag{3.11}$$

When n is even and l is odd, we have

$$C_{ln} = q\,[r^{-1}(b - f\,\mathrm{th}^2 a) + t^{-1}(d\,\mathrm{th}^2 a - b)]\, 2\,\mathrm{cth}\, a\; \rho_l\,(-1)^{1/2(l+1)} \tag{3.12}$$

and finally, when n is odd and l even, we have

$$C_{ln} = q\,[r^{-1}(f\,\mathrm{th}^2 a - c) + t^{-1}(c - d\,\mathrm{th}^2 a)]\, 2\,\mathrm{cth}\, a\; \rho_l\,(-1)^{1/2 l} \tag{3.13}$$

The following notation was used in the above formulas

$$b = 2ak\,\mathrm{th}\, k, \qquad c = 2ak\,\mathrm{cth}\, k, \qquad \rho_l = \tfrac{1}{2}\pi\,(l+1)$$

$$u_n^2 = \mu_n - k^2, \qquad d = (a^2 + \rho_l^2 - u_n^2)\,\mathrm{cth}\, a, \qquad t = d^2 + 4u_n^2 a^2$$

$$f = (a^2 + k^2 + \rho_l^2)\,\mathrm{cth}\, a, \qquad r = f^2 - 4k^2a^2, \qquad q = -a\,\mathrm{csch}\, a.$$

The condition of existence of a nontrivial solution of the homogeneous system (3.5) defines the spectrum of characteristic decrements λ as functions of the parameters of the problem, i.e. of R, a, P numbers and wave number k. The problem of determination of the spectrum is connected with that of obtaining the eigenvalues λ of a normal matrix of the order $Q = N + M$, formed from the coefficients of the system (3.5). The orthogonal step method [2] can be used to reduce the matrix to the quasi-triangular form. This method of finding the eigenvalues of the matrix was used earlier in the investigation of the spectra of perturbations of the plane isothermal flows [3 and 4] and of the convective fluid flow [5]. The Gauss method was used to obtain the eigenvalues of the matrix and all computations were performed on the "Minsk – 2" computer.

4. An approximation containing 20 basis functions ($N = M = 10$) was used to obtain the decrements for $a = 0$ and $a = 3$; for $a = 5$, 24 functions ($N = M = 12$) were used. These approximations yield 10-12 lower levels of the decremental spectrum over the following range of of the Rayleigh numbers $-2000 < R < 4000$ with sufficient accuracy.

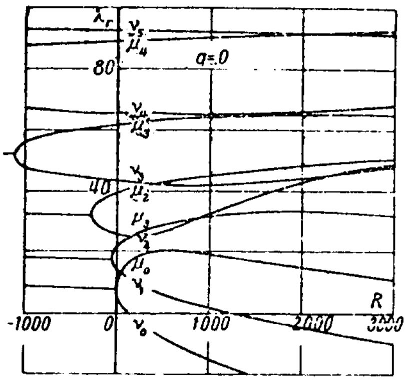

Fig. 1

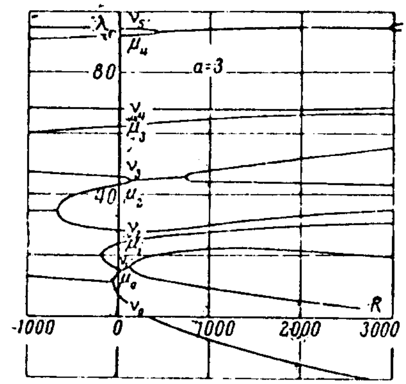

Fig. 2

The convergence of the series (3.1) becomes weaker with increasing R and a. In order to estimate the rate of convergence, we calculated the decrements using a varying number of the basis functions (up to 28). For the Peclet number $a = 3$, the results obtained with the help of 20 and 24 basis functions, practically coincide over the range of values of R quoted above, while for $a = 5$ the approximations containing 24 and 28 basis functions also yielded comparable results.

When the layer was not heated ($R = 0$), then the approximate values of the decrements computed for $a = 3$ and 5, practically coincided with the values which were obtained for this particular case from the exact characteristic relations [6 and 7].

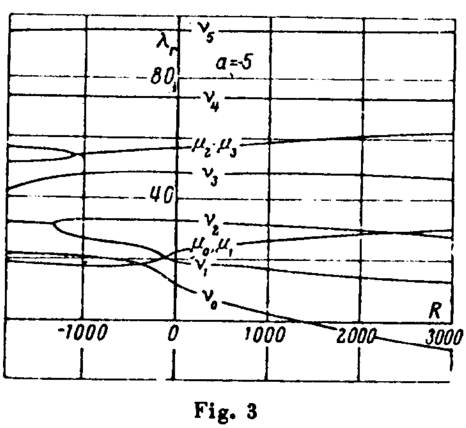

Fig. 3

Let us consider the results obtained. Figs. 1 to 3 show, as an illustration, the relation between the real part λ_r of the perturbation decrement with the wave number $k = 2$ and the Rayleigh number, for the following three values of the Peclet number: $a = 0$, 3 and 5, with $P = 1$. Positive values of R correspond to the heating from below, negative – to the heating from above.

Fig. 1 shows the lower levels of the decremental spectrum for $a = 0$ (no transverse motion: – the Rayleigh problem). In this case we have, for $R = 0$, two types of decaying perturbations present in the flow: isothermic whose spectrum is determined by the boundary value problem (3.2) (μ-levels) and nonisothermic (boundary value problem (3.3), ν – levels).

In the region $R > 0$ all decrements are real (monotonous perturbations) and the spectrum admits only simple intersections such, that the decrements remain real on both sides of the intersection point. Some decrements become negative with increasing R, i.e. monotonous instability takes place.

In the region $R < 0$ real unlike decrements merge, producing complex conjugate pairs, i.e. we have the case of oscillatory perturbations. At the same time the real parts of all decrements remain positive and all perturbations decay.

Figs. 2 and 3 show the decremental spectra in the presence of a transverse fluid flow. We see that injection of the fluid leads to the change in the appearance of the spectrum. Simple intersections disappear at arbitrarily small values of a and, either two real levels merge into a complex conjugate pair (on increasing the Rayleigh number these pairs separate back into real levels, see Fig. 4), or real levels diverge without intersection..With increasing a the points of intersection of the decrement lines with the R-axis are displaced into the region of high values of R, and this indicates that the inward flow opposes the tendency towards convective instability (*).

In the region $R < 0$, increase in a is accompanied by displacement of the points, at which the real levels merge towards the high absolute values of R.

We note that it follows directly from (2.3) that the change in the sign of a, i.e. reversing the direction of the flow v_0 does not alter the decremental spectra. The amplitudes however, are changed thus: $v(z) \to v(-z)$ and $\theta(z) \to \theta(-z)$. It follows, that, when $a > 0$, the perturbations are situated near the upper boundary, while when $a < 0$ – near the lower boundary.

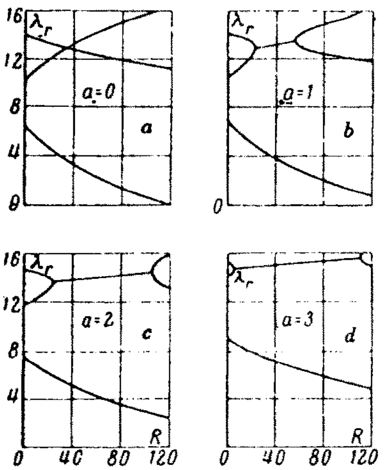

Fig. 4

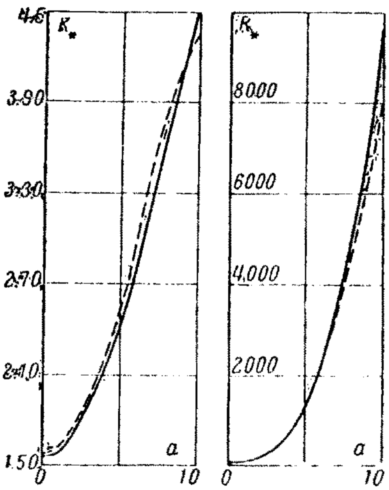

Fig. 5

When investigating the convective instability of a plane layer with permeable boundaries we find, that the lowest level responsible for the onset of instability merits most attention. Numerical data obtained in computing the decremental spectra allow us to construct (for fixed a and P) a relationship between the critical Rayleigh numbers and the perturbation wave number k, i.e. the neutral curve of monotonous perturbations. This curve has a minimum at $k = k_*$; we shall denote the corresponding critical Rayleigh number defining the boundary of the monotonous instability by R_*.

The data available from the computation of spectra and of the neutral curves make it possible to find the relationship between R_*, k_*, and the Peclet number. Fig. 5 shows these

*) Similar effect of stabilizing the Rayleigh type instability occurs, when one of the planes bounding the horizontal fluid layer, moves (see [8]).

relationships for $P = 1$. Solid lines show the results obtained using 16 basis functions, and broken lines – using 8 basis functions.

We see that the transverse flow has a strong influence on the convective stability of the fluid increasing it appreciably. Increase in the values of R_* and k_* as the Peclet number increases, is physically understandable. Indeed, the transverse fluid flow displaces the perturbations towards the boundary and they become localized within a region whose height decreases as the Peclet number increases. This, of course, results in the value of R_* increasing together with that of the Peclet number.

Decrease in the vertical dimension of the region of actual formation of perturbations, is accompanied by a decrease in the wavelength of the most dangerous perturbations, i.e. by an increase in the value of k_*.

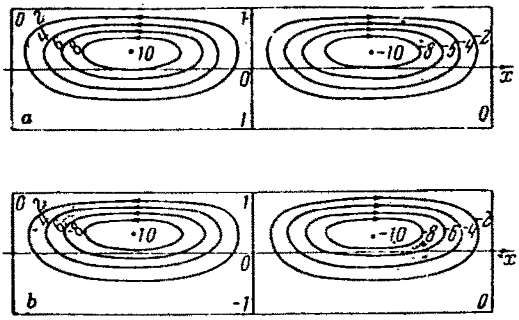

Fig. 6

Fig. 6 and 7 show the streamlines of characteristic perturbations whose wave numbers are $k = 2$, $k_2 = 0$ when $a = 3$ and $P = 1$. The flow pattern shown, occurs over the interval of the x-coordinate equal to the perturbation wavelength.

Fig. 6 corresponds to the decaying monotonous perturbations with the decrement μ_0 (Fig. 6*a*) and ν_1 (Fig. 6*b*) up to their merger at $R = 5$ (the corresponding values of λ are 14.176 and 16.000).

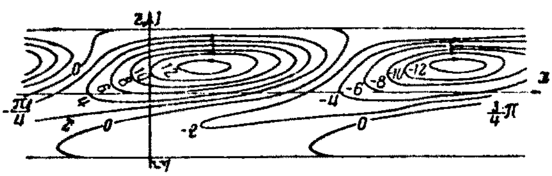

Fig. 7

Fig. 7 shows the streamlines of decaying oscillatory perturbations with the decrement $\lambda = 20.257 + 9.4376i$ for a pair of merging levels μ_0 and ν_0 at $R = -1200$.

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